THEORY GUIDE

Oblique Shock Web Application

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Atkinson Science welcomes your comments on this Theory Guide. Please send an email to keith.atkinson@atkinsonscience.co.uk.

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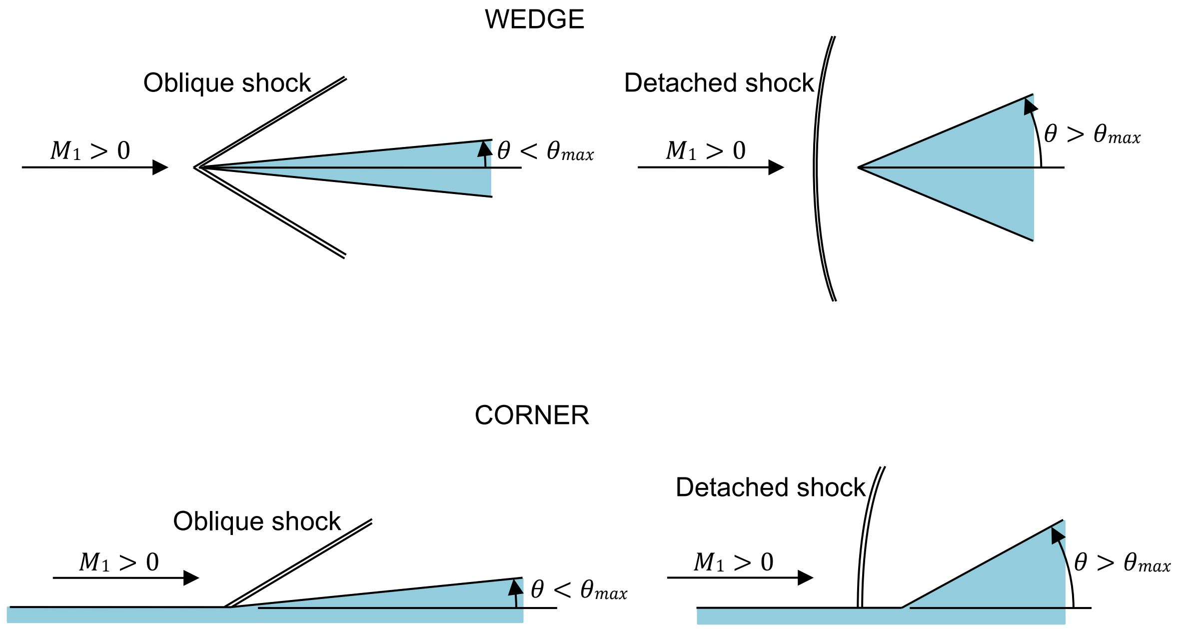
1 Introduction

You can find the Atkinson Science Oblique Shock web application at the web address https://atkinsonscience.co.uk/WebApps/Aerospace/ObliqueShock.aspx. The application calculates the change in properties across a steady, two-dimensional oblique shock. The equations that relate the conditions on the two sides of an oblique shock can be found in text books on compressible flow, such as Refs. [1] and [2]. The application was developed for use with the International Standard Atmosphere, Ref. [3], for which the ratio of specific heats γ is defined to be 1.4.

Suppose a steady uniform supersonic flow with Mach number M_1 approaches a wedge or corner which has a *deflection angle* of θ, as shown in Figure 1. The oblique shock equations show that for any M_1 there is a maximum deflection angle θ_{max}. If $\theta < \theta_{max}$ then there are two possible oblique shock solutions: a *weak shock* and a *strong shock*. If a weak shock is formed then the Mach number M_2 after the shock is less than M_1 but remains greater than 1 (supersonic flow). If a strong shock is formed then M_2 is less than 1 (subsonic flow), and the rise in pressure, density and temperature across the shock is more severe. In both cases the shock is *attached* to the wedge or corner, as shown in Figure 1. The *wave angle* β of the weak shock is less than that of the strong shock, but they become the same as θ approaches θ_{max}.

If $\theta > \theta_{max}$ then a *detached shock* is formed, as shown in Figure 1. The Oblique Shock web application does not deal with the detached shock and will inform the user when $\theta > \theta_{max}$.

Figure 1 Oblique (attached) shock and detached shock



2 Upstream properties

The properties of the flow upstream of the oblique shock are based on the International Standard Atmosphere (ISA), Ref. [3]. The ISA is a model of the change in temperature and pressure with altitude in the Earth's atmosphere. The atmosphere is divided into layers over which the temperature is either constant or varies linearly with geopotential altitude, as shown in Figure 2. Ref. [3] defines a number of constants and formulae by which the pressure and other properties of the atmosphere may be calculated from the temperature. The constants are set out in Table 1.

Figure 2 Temperature of the International Standard Atmosphere

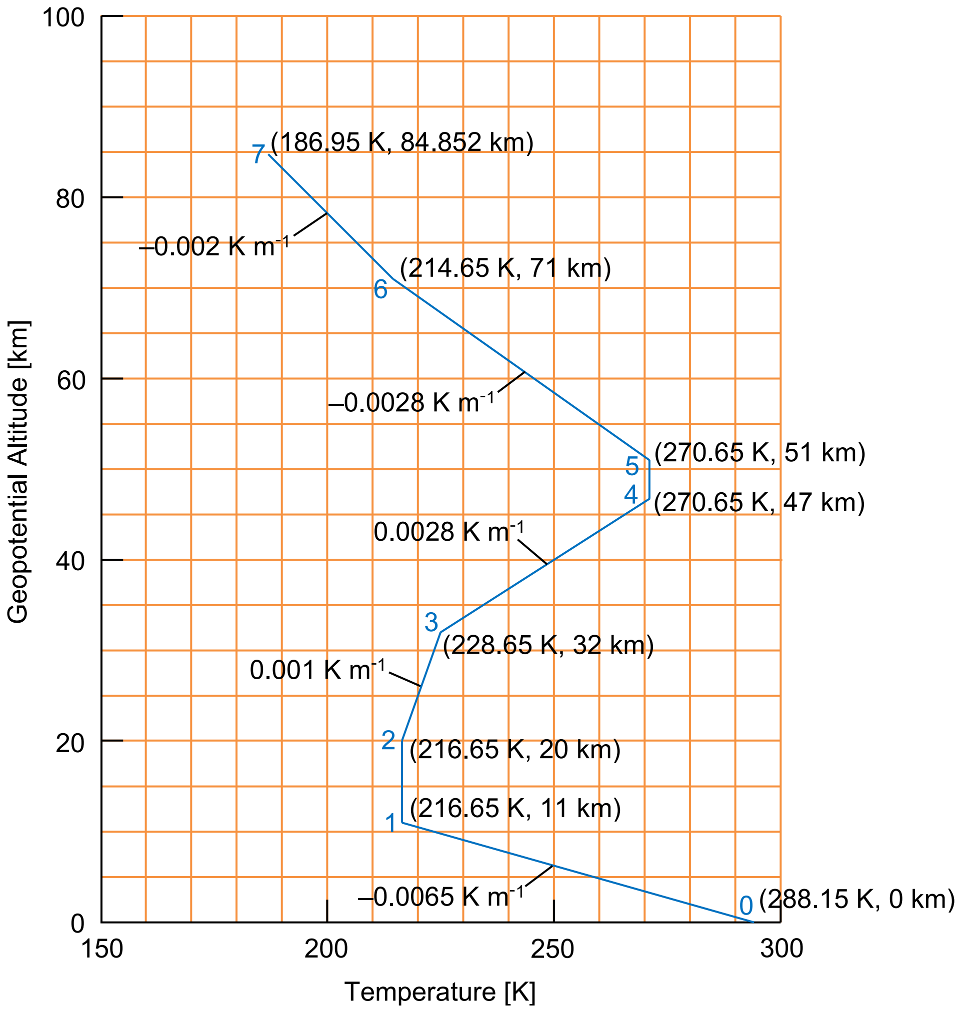


Table 1 Properties of the International Standard Atmosphere

Standard values at sea level	
Temperature T	288.15 K
Pressure p	101,325 Pa
Density ρ	1.2250 kg m^{-3}
Dynamic viscosity μ	1.7894 × 10^{-5} kg m^{-1} s^{-1}
Speed of sound a	340.29 m s^{-1}
Acceleration due to gravity g	9.80665 m s^{-2}
Other standard values	
Specific gas constant of air R_{Air}	287.05287 J kg^{-1} K^{-1}
Ratio of specific heats $\gamma = c_p/c_v$	1.4

The geopotential altitude h is related to the geometric altitude z by

$$h(z) = \left(\frac{R_E}{R_E + z}\right) z$$

where R_E is the radius of the Earth (6,356 km). By rearranging this equation we can write the geometric altitude z in terms of the geopotential altitude h:

$$z(h) = \left(\frac{R_E}{R_E - h}\right) h$$

Ref. [3] shows how the atmospheric pressure and density are calculated given the variation in atmospheric temperature with altitude. The calculation steps are also given in Ref. [4]. Table 2 gives the pressure and density at the points 0 to 7 in Figure 2.

Table 2 Pressure and density of the International Standard Atmosphere

Point	Geopotential altitude h [m]	Geometric altitude z [m]	Temperature T [K]	Pressure p [Pa]	Density ρ [kg m^{-3}]
0	0	0	288.15	101,325	1.2250
1	11,000	11,109	216.65	22,632	0.3639
2	20,000	20,063	216.65	5,475	0.08804
3	32,000	32.162	228.65	868.0	0.01322
4	47,000	47.350	270.65	110.9	0.001427
5	51,000	51,413	270.65	66.94	8.616 × 10^{-4}
6	71,000	71,802	214.65	3.956	6.421 × 10^{-5}
7	84,852	86,000	186.95	0.3734	6.958 × 10^{-6}

The speed of sound a is given by

$$a = \sqrt{\gamma R_{Air} T}$$

where the ratio of the specific heats $\gamma = c_p/c_v$ is defined to be constant and equal to 1.4 and the specific gas constant of the air R_{Air} is defined to be 287.05287 J kg^{-1}K^{-1} (see Table 1).

The Mach number M is

$$M = \frac{w}{a}$$

where w is the air speed.

The equations that relate the conditions on the two sides of an oblique shock assume that the air is calorically perfect. The static enthalpy is then equal to $c_p\,T$, where the specific heat at constant pressure c_p is constant. In the web application c_p is taken to be 1.004 kJ kg^{-1} K^{-1}, which is the specific heat at constant pressure of dry air at 15°C.

If the air is assumed to be a calorically perfect gas, then the entropy of the air is given by

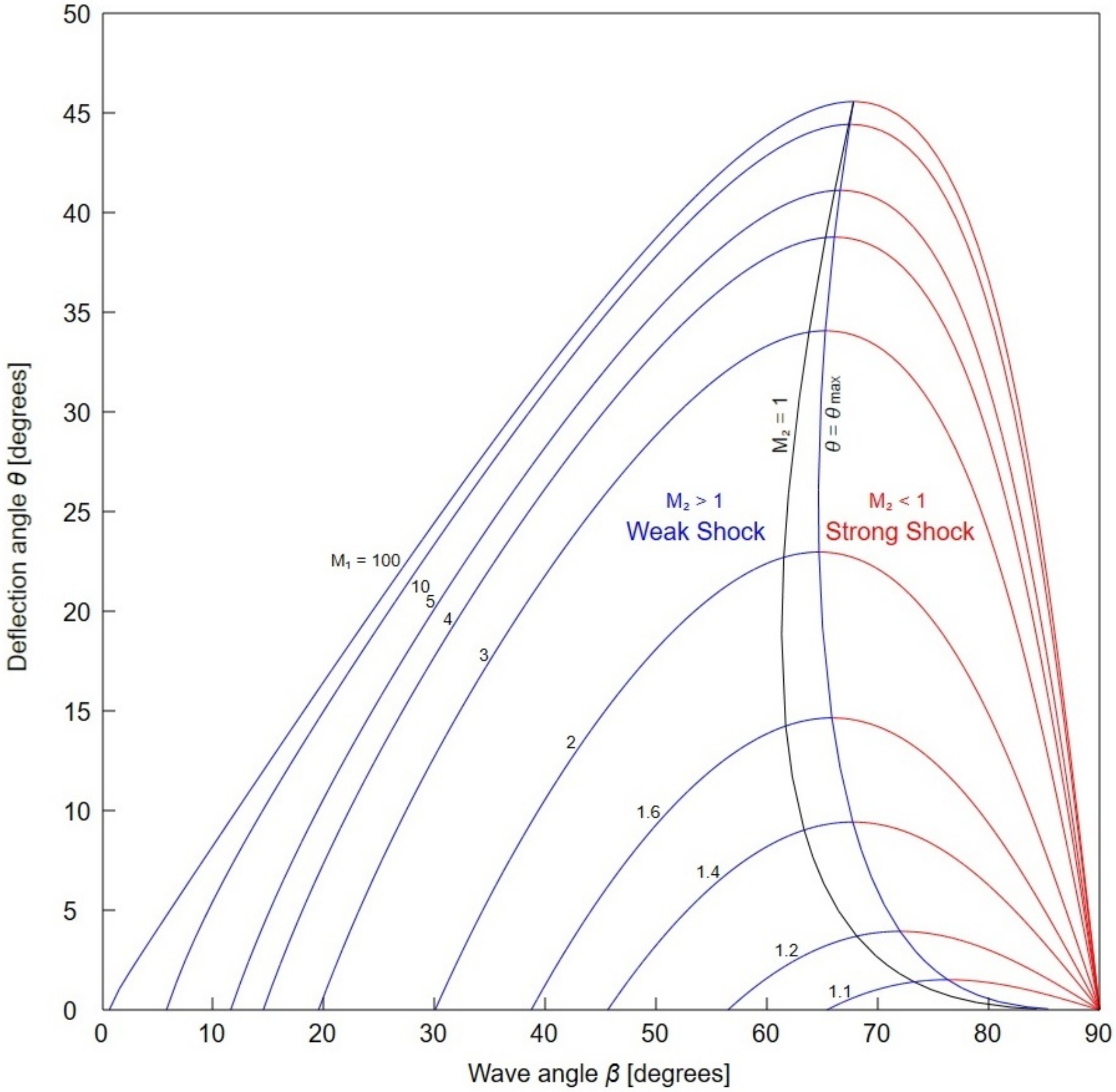
$$s - s_o = c_p \ln(T/T_o) - R_{Air} \ln(p/p_o)$$

where (p_o, T_o, s_o) are conditions at some reference state. Referring to the tables of thermodynamic properties, Ref. [5], we have taken s_o to be 6.86305 kJ kg^{-1} K^{-1} at 15°C and 1 bar pressure.

3 Oblique shock solutions

Figure 3 shows all the possible oblique shock solutions. One can see that for any upstream Mach number M_1 and any deflection angle $\theta < \theta_{max}$, there are two possible solutions with different wave angles β. In the strong shock solution the flow becomes subsonic. In the weak shock solution the flow remains supersonic, except over a small range of θ slightly smaller than θ_{max}.

Figure 3 Oblique shock solutions

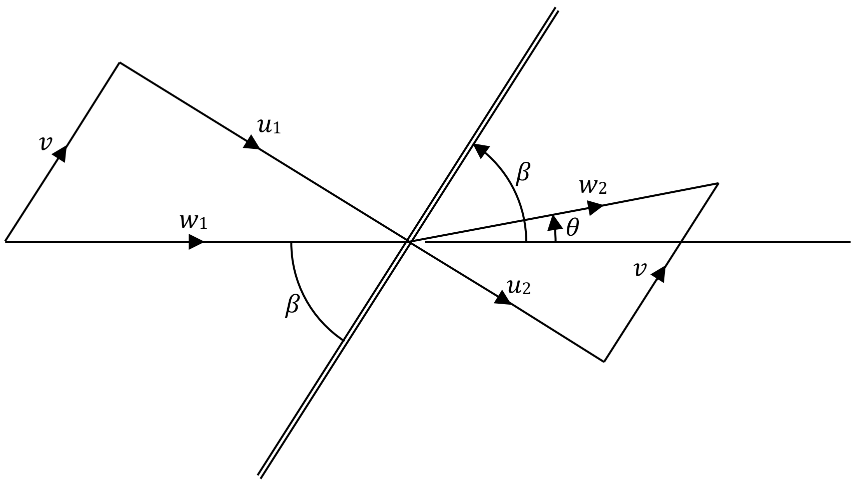


4 Oblique shock equations

4.1 Velocity components

Figure 4 shows the velocity triangles for an oblique shock wave. The shock wave is indicated by a double line.

Figure 4 Velocity triangles for an oblique shock wave



The upstream Mach number M_1 is w_1/a_1, where a_1 is the speed of sound upstream of the shock. The velocity component normal to the shock on the upstream side, u_1, is given by

$$u_1 = w_1 \sin\beta$$

or

$$\frac{u_1}{a_1} = M_1 \sin\beta$$

Similarly, the downstream Mach number M_2 is w_2/a_2, where a_2 is the speed of sound downstream of the shock. The velocity component normal to the shock on the downstream side, u_2, is given by

$$u_2 = w_2 \sin(\beta - \theta)$$

or

$$\frac{u_2}{a_2} = M_2 \sin(\beta - \theta)$$

The component of velocity parallel to the shock, v, is the same on both sides.

4.2 Thermodynamic properties

Equations for the change in thermodynamic properties across the shock are derived in Refs [1] and [2]. They are:

Density

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)M_1^2\sin^2\beta}{(\gamma-1)M_1^2\sin^2\beta + 2}$$

Pressure

$$\frac{p_2 - p_1}{p_1} = \frac{2\gamma}{\gamma+1}(M_1^2\sin^2\beta - 1)$$

Temperature, static enthalpy and speed of sound

$$\frac{T_2}{T_1} = \frac{h_2}{h_1} = \frac{a_2^2}{a_1^2} = 1 + \frac{2(\gamma-1)}{(\gamma+1)^2}\,\frac{(M_1^2\sin^2\beta - 1)}{(M_1^2\sin^2\beta)}\,(\gamma M_1^2\sin^2\beta + 1)$$

Entropy

$$\frac{s_2 - s_1}{R} = \ln\left\{\left[1 + \frac{2}{\gamma-1}(M_1^2\sin^2\beta - 1)\right]^{1/(\gamma-1)}\left[\frac{(\gamma+1)M_1^2\sin^2\beta}{(\gamma-1)M_1^2\sin^2\beta + 2}\right]^{-\gamma/(\gamma-1)}\right\}$$

4.3 Relation between the wave angle and the deflection angle

Referring to Figure 4, we have

$$\tan\beta = \frac{u_1}{v}$$

and

$$\tan(\beta - \theta) = \frac{u_2}{v}$$

After eliminating v,

$$\frac{\tan(\beta-\theta)}{\tan\beta} = \frac{u_2}{u_1}$$

Using continuity ($\rho_1 u_1 = \rho_2 u_2$) and the density equation gives

$$\frac{\tan(\beta-\theta)}{\tan\beta} = \frac{\rho_1}{\rho_2} = \frac{(\gamma-1)M_1^2\sin^2\beta + 2}{(\gamma+1)M_1^2\sin^2\beta}$$

This is an implicit relation between the wave angle β and the deflection angle θ for a given upstream Mach number M_1. By manipulating this equation we can find an explicit equation for θ:

$$\tan\theta = 2\cot\beta\,\frac{M_1^2\sin^2\beta - 1}{M_1^2(\gamma + \cos 2\beta) + 2}$$

Finding an equation for the wave angle β in terms of the deflection angle θ and M_1 is more difficult. We have used the procedure in Ref. [6]. By squaring both sides of the preceding equation we can obtain a cubic equation in terms of $\sin^2\beta$.

$$\tan^2\theta = 4\cot^2\beta\,\frac{[M_1^2\sin^2\beta - 1]^2}{[M_1^2(\gamma + \cos 2\beta) + 2]^2}$$

$$= \frac{4\cos^2\beta}{\sin^2\beta} \times \frac{[M_1^2\sin^2\beta - 1]^2}{[M_1^2(\gamma + \cos^2\beta - \sin^2\beta) + 2]^2}$$

$$= \frac{4(1 - \sin^2\beta)}{\sin^2\beta} \times \frac{[M_1^2\sin^2\beta - 1]^2}{[M_1^2(\gamma + 1 - 2\sin^2\beta) + 2]^2}$$

Replacing $\sin^2\beta$ with X,

$$\tan^2\theta = \frac{4(1 - X)}{X} \times \frac{[M_1^2 X - 1]^2}{[M_1^2(\gamma + 1 - 2X) + 2]^2}$$

By cross-multiplying and separating terms in X^3, X^2, and X, we obtain

$$X^3 - X^2\left[\frac{M_1^2 + 2}{M_1^2} + \gamma\sin^2\theta\right] + X\left[\frac{2M_1^2 + 1}{M_1^4} + \left(\frac{\gamma + 1}{2}\right)^2 \sin^2\theta + \frac{(\gamma - 1)}{M_1^2}\sin^2\theta\right] - \frac{\cos^2\theta}{M_1^4} = 0$$

or

$$AX^3 + BX^2 + CX + D = 0$$

where

$$A = 1$$

$$B = -\left[\frac{M_1^2 + 2}{M_1^2} + \gamma\sin^2\theta\right]$$

$$C = \left[\frac{2M_1^2 + 1}{M_1^4} + \left(\frac{\gamma + 1}{2}\right)^2 \sin^2\theta + \frac{(\gamma - 1)}{M_1^2}\sin^2\theta\right]$$

$$D = -\frac{\cos^2\theta}{M_1^4}$$

To determine the roots of the cubic equation, we define

$$Q = \frac{3C - B^2}{9}$$

$$R = \frac{9BC - 27D - 2B^3}{54}$$

$$D = Q^3 + R^2$$

There are no solutions for an oblique shock if $D \geq 0$. A value of $D \geq 0$ indicates that $\theta \geq \theta_{max}$ and the shock is detached. When the web application detects that $D \geq 0$, it issues a message asking the user to either reduce the deflection angle θ or increase the Mach number M_1.

If $D < 0$ then the wave angle of the weak shock β_w is

$$\beta_w = \tan^{-1}\left(\sqrt{\frac{X_w}{1 - X_w}}\right)$$

and the wave angle of the strong shock β_s is

$$\beta_s = \tan^{-1}\left(\sqrt{\frac{X_s}{1 - X_s}}\right)$$

where

$$X_w = -\frac{B}{3} - \sqrt{-Q}\left(\cos\emptyset - \sqrt{3}\sin\emptyset\right),$$

$$X_s = -\frac{B}{3} + 2\sqrt{-Q}\cos\emptyset,$$

$$\emptyset = \frac{1}{3}\left[\tan^{-1}\left(\frac{\sqrt{-D}}{R}\right) + \Delta\right]$$

and

$$\Delta = \begin{cases} 0 \text{ if } R \geq 0 \\ \pi \text{ if } R < 0 \end{cases}$$

5 References

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5. C. Borgnakke and R. E. Sonntag, *Thermodynamic and Transport Properties*, Wiley, 1997.
6. T. T. Hartley, R. Brandis and F. Mossayebi, Exact and Approximate Solutions to the Oblique Shock Equations for Real-Time Applications, NASA CR 187173, 1991.